

# On a family of differential approximations of the radiative transfer equation

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Received: 29 July 2011 / Accepted: 12 September 2011 / Published online: 1 October 2011  
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**Abstract** The radiative transfer equation (RTE) arises in a variety of applications and is challenging to solve numerically due to its integro-differential form and high dimension. For highly forward-peaked media, it is even more difficult to solve RTE since accurate numerical solutions require a high resolution of the direction variable. For this reason, various approximations of RTE have been proposed in the literature. In this paper, we study a family of differential approximations of the RTE in three spatial variables. We explain the idea of constructing the differential approximations, and comment on the usefulness of the approximations.

**Keywords** Radiative transfer equation · Highly forward-peaked scattering · Generalized Fokker–Planck equation · Differential approximation

## 1 Introduction

The radiative transfer equation (RTE) arises in a variety of applications of physics, chemistry, and other areas of sciences and engineering; see, e.g.,

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[1, 4, 6–8, 18, 19, 23, 24]. Recently, there has been much research interest in inverse problems related to RTE in biomedical imaging applications (e.g., [2, 3, 15]).

RTE is a high dimensional problem and is challenging to solve numerically ([11]). In many applications, the light propagation is highly forward-peaked. For highly forward-peaked media, it is even more difficult to solve RTE since accurate numerical solutions require a high resolution of the direction variable. For this reason, various approximations of RTE have been proposed in the literature, e.g., the delta-Eddington approximation [13], the Fokker–Planck approximation [20, 21], the Boltzmann–Fokker–Planck approximation [22, 5], the generalized Fokker–Planck approximation [16], the Fokker–Planck–Eddington approximation and the generalized Fokker–Planck–Eddington approximation [10]. For RTE with high absorption and small geometries, the simplified spherical harmonics ( $SP_N$ ) method ([14]) is shown to produce good approximate solutions ([9]). Well-posedness of the ( $SP_N$ ) method is shown in [25]. In this paper, we provide a preliminary study of a family of differential approximations of RTE.

The approximations are similar to some generalized Fokker–Planck equations studied in [16]. However, one purpose of this paper is to take a first step in establishing a framework for development of a sequence of differential operators that converges to the integral operator of RTE. The approximations discussed in this paper automatically mimic the limiting behavior of the eigenvalues of the integral operator of the RTE. More precisely, for the eigenvalues  $\{k_n\}$  of the integral operator and the eigenvalues  $\{k_n^{(a)}\}$  of an approximation operator, we have

$$\lim_{n \rightarrow \infty} k_n^{(a)} = \lim_{n \rightarrow \infty} k_n = 0.$$

In addition, the first several eigenvalues for the integral operator and the approximation operators are pairwise equal.

After a review of RTE in Sect. 2, we introduce a general framework for the family of differential approximation models in Sect. 3. In Sect. 4, we provide a detailed discussion on conditions for positivity of the parameters in the one-term and two-term approximation models. In Sect. 5, we consider the approximation models for the case where the Henyey–Greenstein phase function is used. In the final section, some concluding remarks are given.

## 2 The radiative transfer equation

Let  $X$  be a domain in  $\mathbb{R}^3$  with a Lipschitz boundary  $\partial X$ . Denote by  $\Omega$  the unit sphere in  $\mathbb{R}^3$ . Let  $\Gamma_- \subset \partial X \times \Omega$  be the incoming boundary. Denote by  $d\sigma(\omega)$  the infinitesimal area element on the unit sphere  $\Omega$ . If we introduce the spherical coordinate system by

$$\omega = (\sin \theta \cos \psi, \sin \theta \sin \psi, \cos \theta)^T, \quad 0 \leq \theta \leq \pi, \quad 0 \leq \psi \leq 2\pi, \quad (2.1)$$

then  $d\sigma(\omega) = \sin \theta \, d\theta \, d\psi$ . We will need an integral operator  $S$  defined by

$$(Su)(x, \omega) = \int_{\Omega} k(\omega \cdot \hat{\omega}) u(x, \hat{\omega}) \, d\sigma(\hat{\omega}) \quad (2.2)$$

with  $k$  a nonnegative normalized phase function:

$$\int_{\Omega} k(\boldsymbol{\omega} \cdot \hat{\boldsymbol{\omega}}) d\sigma(\hat{\boldsymbol{\omega}}) = 1 \quad \forall \boldsymbol{\omega} \in \Omega. \tag{2.3}$$

One well-known example is the Henyey–Greenstein phase function (cf. [12])

$$k(t) = \frac{1 - g^2}{4\pi(1 + g^2 - 2gt)^{3/2}}, \quad t \in [-1, 1], \tag{2.4}$$

where the parameter  $g \in (-1, 1)$  is the anisotropy factor of the scattering medium. Note that  $g = 0$  for isotropic scattering,  $g > 0$  for forward scattering, and  $g < 0$  for backward scattering.

With the above notation, a boundary value problem of the RTE reads (cf. [1, 17])

$$\boldsymbol{\omega} \cdot \nabla u(\mathbf{x}, \boldsymbol{\omega}) + \mu_t(\mathbf{x}) u(\mathbf{x}, \boldsymbol{\omega}) = \mu_s(\mathbf{x}) (Su)(\mathbf{x}, \boldsymbol{\omega}) + f(\mathbf{x}, \boldsymbol{\omega}), \tag{2.5}$$

$$(\mathbf{x}, \boldsymbol{\omega}) \in X \times \Omega,$$

$$u(\mathbf{x}, \boldsymbol{\omega}) = 0, \quad (\mathbf{x}, \boldsymbol{\omega}) \in \Gamma_-. \tag{2.6}$$

Here  $\mu_t = \mu_a + \mu_s$ ,  $\mu_a$  is the macroscopic absorption cross section,  $\mu_s$  is the macroscopic scattering cross section, and  $f$  is a source function. We assume these given functions satisfy

$$\mu_t, \mu_s \in L^\infty(X), \quad \mu_s \geq 0 \text{ a.e. in } X, \tag{2.7}$$

$$\mu_t - \mu_s \geq c_0 \text{ in } X \text{ for some constant } c_0 > 0,$$

$$f \in L^2(X \times \Omega). \tag{2.8}$$

These assumptions are naturally valid in applications; the last part of (2.7) means that the absorption effect is not negligible. The homogeneous boundary condition (2.6) corresponds to a vacuum setting around  $X$ . It is equally well to consider a general incoming flux boundary condition  $u(\mathbf{x}, \boldsymbol{\omega}) = u_{\text{in}}(\mathbf{x}, \boldsymbol{\omega})$  on  $\Gamma_-$  with a given function  $u_{\text{in}}$ .

It is shown in [1] that the problem (2.5)–(2.6) has a unique solution  $u \in H_2^1(X \times \Omega)$ , where

$$H_2^1(X \times \Omega) := \{v \in L^2(X \times \Omega) \mid \boldsymbol{\omega} \cdot \nabla v \in L^2(X \times \Omega)\}$$

with  $\boldsymbol{\omega} \cdot \nabla v$  denoting the generalized directional derivative of  $v$  in the direction  $\boldsymbol{\omega}$  (cf. [1]).

### 3 Approximation

For a spherical harmonic of order  $n$ ,  $Y_n(\boldsymbol{\omega})$ ,

$$(SY_n)(\boldsymbol{\omega}) = k_n Y_n(\boldsymbol{\omega}), \tag{3.1}$$

where

$$k_n = 2\pi \int_{-1}^1 k(s) P_n(s) ds, \quad P_n : \text{the Legendre polynomial of degree } n. \quad (3.2)$$

The relation (3.1) states that  $k_n$  is an eigenvalue of  $S$  with spherical harmonics of order  $n$  as eigenfunctions. Since

$$\|P_n\|_{L^2(-1,1)} = \sqrt{\frac{2}{2n+1}},$$

we have

$$|k_n| \leq 2\pi \|k\|_{L^2(-1,1)} \|P_n\|_{L^2(-1,1)} = 2\pi \sqrt{\frac{2}{2n+1}} \|k\|_{L^2(-1,1)}.$$

So

$$\{k_n\} \text{ is bounded and } k_n \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (3.3)$$

Denote by  $\Delta^*$  the Laplace-Beltrami operator on the unit sphere  $\Omega$ . Then,

$$-(\Delta^* Y_n)(\omega) = n(n+1) Y_n(\omega). \quad (3.4)$$

Let  $\{Y_{n,m} \mid -n \leq m \leq n, n \geq 0\}$  be an orthonormalized basis in  $L^2(\Omega)$ . We have the expansion

$$u(\omega) = \sum_{n=0}^{\infty} \sum_{m=-n}^n u_{n,m} Y_{n,m}(\omega) \text{ in } L^2(\Omega), \quad u_{n,m} = \int_{\Omega} u(\omega) Y_{n,m}(\omega) d\sigma(\omega). \quad (3.5)$$

With such an expansion of  $u \in L^2(\Omega)$ , we have an expansion for  $Su$ :

$$Su(\omega) = \sum_{n=0}^{\infty} k_n \sum_{m=-n}^n u_{n,m} Y_{n,m}(\omega) \text{ in } L^2(\Omega). \quad (3.6)$$

Now suppose there are real numbers  $\{\lambda_i, \alpha_i\}_{i \geq 1}$  such that

$$\sum_{i=1}^{\infty} \frac{\lambda_i}{1+n(n+1)\alpha_i} = k_n, \quad n = 0, 1, \dots \quad (3.7)$$

Then formally, we have

$$\begin{aligned} \sum_{i=1}^{\infty} \lambda_i (I - \alpha_i \Delta^*)^{-1} u(\omega) &= \sum_{n=0}^{\infty} \sum_{m=-n}^n u_{n,m} \sum_{i=1}^{\infty} \frac{\lambda_i}{1 + n(n+1)\alpha_i} Y_{n,m}(\omega) \\ &= \sum_{n=0}^{\infty} k_n \sum_{m=-n}^n u_{n,m} Y_{n,m}(\omega) \\ &= Su(\omega). \end{aligned}$$

In other words,

$$S = \sum_{i=1}^{\infty} \lambda_i (I - \alpha_i \Delta^*)^{-1}. \tag{3.8}$$

With  $n = 0$  in (3.7),

$$\sum_{i=1}^{\infty} \lambda_i = 1;$$

consequently, a necessary condition for (3.7) is

$$\lambda_i \rightarrow 0 \text{ as } i \rightarrow \infty.$$

The formal equality (3.8) motivates us to consider approximating  $S$  by operators of the following form

$$S_j = \sum_{i=1}^j \lambda_{j,i} (I - \alpha_{j,i} \Delta^*)^{-1}. \tag{3.9}$$

The eigenvalues of  $S_j$  are

$$\sum_{i=1}^j \lambda_{j,i} (1 + n(n+1)\alpha_{j,i})^{-1}$$

with associated eigenfunctions the spherical harmonics of order  $n$ :

$$(S_j Y_n)(\omega) = \left[ \sum_{i=1}^j \frac{\lambda_{j,i}}{1 + n(n+1)\alpha_{j,i}} \right] Y_n(\omega).$$

Note that for fixed  $j$ ,

$$\sum_{i=1}^j \frac{\lambda_{j,i}}{1+n(n+1)\alpha_{j,i}} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Thus, the eigenvalue sequence of  $S_j$  has a unique accumulation point 0, a property shared by the operator  $S$ , cf. (3.3). Hence, we choose the parameters  $\{\lambda_{j,i}, \alpha_{j,i}\}_{i=1}^j$  so that for some integer  $n_j$  depending on  $j$ ,

$$\sum_{i=1}^j \frac{\lambda_{j,i}}{1+n(n+1)\alpha_{j,i}} = k_n, \quad n = 0, 1, \dots, n_j - 1. \tag{3.10}$$

We require

$$n_j \rightarrow \infty \text{ as } j \rightarrow \infty. \tag{3.11}$$

**Theorem 3.1** Assume (3.10) holds and

$$\sup_{n \geq n_j} \left| \sum_{i=1}^j \lambda_{j,i} (1+n(n+1)\alpha_{j,i})^{-1} \right| \rightarrow 0 \text{ as } j \rightarrow \infty. \tag{3.12}$$

Then

$$\|S_j - S\|_{\mathcal{L}(L^2(\Omega), L^2(\Omega))} \rightarrow 0 \text{ as } j \rightarrow \infty. \tag{3.13}$$

*Proof* For any  $u \in L^2(\Omega)$ , with the expansion (3.5), we have

$$(Su - S_j u)(\omega) = \sum_{n=n_j}^{\infty} \sum_{m=-n}^n u_{n,m} \left( k_n - \sum_{i=1}^j \lambda_{j,i} (1+n(n+1)\alpha_{j,i})^{-1} \right) Y_{n,m}(\omega).$$

Thus,

$$\begin{aligned} \|Su - S_j u\|_0^2 &= \sum_{n=n_j}^{\infty} \sum_{m=-n}^n |u_{n,m}|^2 \left| k_n - \sum_{i=1}^j \lambda_{j,i} (1+n(n+1)\alpha_{j,i})^{-1} \right|^2 \\ &\leq 4 \sup_{n \geq n_j} \left\{ |k_n|, \left| \sum_{i=1}^j \lambda_{j,i} (1+n(n+1)\alpha_{j,i})^{-1} \right| \right\}^2 \|u\|_0^2. \end{aligned}$$

So we have the convergence (3.13) by (3.3) and (3.12). □

A sufficient condition for (3.12) is that all  $\lambda_{j,i}$  and  $\alpha_{j,i}$  are positive.

**Theorem 3.2** Assume (3.10) and  $\lambda_{j,i} > 0$  and  $\alpha_{j,i} > 0$  for  $i = 1, \dots, j$ . Then (3.12) holds.

*Proof* Since all  $\lambda_{j,i}$  and  $\alpha_{j,i}$  are positive, for  $n \geq n_j$ ,

$$\left| \sum_{i=1}^j \lambda_{j,i} (1 + n(n+1)\alpha_{j,i})^{-1} \right| \leq \sum_{i=1}^j \lambda_{j,i} (1 + (n_j - 1)n_j\alpha_{j,i})^{-1} = k_{n_j} \rightarrow 0 \text{ as } j \rightarrow \infty.$$

Thus, (3.12) holds. □

Notice that  $\alpha_{j,i} > 0$  is needed to ensure ellipticity of the differential operator  $(I - \alpha_{j,i}\Delta^*)$ . When we discretize the operator  $S_j$ , the positivity of  $\{\lambda_{j,i}\}_{i=1}^j$  is desirable for numerical stability in computing approximations of  $S_j$ .

Regarding the condition (3.11) for the system (3.10), let us show that  $n_j \geq j - 1$ . Indeed, we have the following result.

**Proposition 3.3** There exist pairwise distinct positive numbers  $\alpha_1, \dots, \alpha_j$  such that the system (3.10) with  $n_j = j - 1$  is uniquely solvable for  $\lambda_1, \dots, \lambda_j$ .

*Proof* We use mathematical induction. The result is obvious for  $j = 1$ . Assume the statement holds for  $j = k$ . Then the determinant

$$\Delta_k = \begin{vmatrix} 1 & 1 & \dots & 1 \\ \frac{1}{1+2\alpha_1} & \frac{1}{1+2\alpha_2} & \dots & \frac{1}{1+2\alpha_k} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{1+(k-1)k\alpha_1} & \frac{1}{1+(k-1)k\alpha_2} & \dots & \frac{1}{1+(k-1)k\alpha_k} \end{vmatrix} \neq 0.$$

For a real variable  $\alpha$ , consider the function

$$\Delta_{k+1}(\alpha) = \begin{vmatrix} 1 & 1 & \dots & 1 & 1 \\ \frac{1}{1+2\alpha_1} & \frac{1}{1+2\alpha_2} & \dots & \frac{1}{1+2\alpha_k} & \frac{1}{1+2\alpha} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{1+k(k+1)\alpha_1} & \frac{1}{1+k(k+1)\alpha_2} & \dots & \frac{1}{1+k(k+1)\alpha_k} & \frac{1}{1+k(k+1)\alpha} \end{vmatrix}.$$

Obviously,  $\Delta_{k+1}(\alpha)$  is a rational function of  $\alpha$ . Since

$$(1 + k(k+1)\alpha) \Delta_{k+1}(\alpha) \rightarrow \Delta_k \neq 0 \text{ as } \alpha \rightarrow -1/(k(k+1)),$$

the rational function  $\Delta_{k+1}(\alpha)$  is not identically zero. Therefore,  $\Delta_{k+1}(\alpha) \neq 0$  except for finite number of values of  $\alpha$  and the statement is true for  $j = k + 1$ . □

Note that in the context of the proof of Theorem 3.3 with arbitrary choices of  $\alpha_{j,i} > 0, 1 \leq i \leq j$ , there is no guarantee for  $\lambda_{j,i} > 0, 1 \leq i \leq j$ .

#### 4 One-term and two-term approximations

We use an operator  $S_j$  of the form (3.9) to approximate  $S$ . From now on, we drop the letter  $j$  in the subscripts for  $\lambda_{j,i}$  and  $\alpha_{j,i}$ . As noted after Theorem 3.2, to maintain ellipticity of the differential operator  $(I - \alpha_i \Delta^*)$ , we require

$$\alpha_i > 0, \quad 1 \leq i \leq j. \quad (4.1)$$

Moreover, for stable numerical approximation of the operator  $S_j$ , it is desirable to have

$$\lambda_i > 0, \quad 1 \leq i \leq j, \quad (4.2)$$

wherever possible. For the numbers  $\{k_n\}$  defined in (3.2), recall the property (3.3) and assume

$$k_0 \geq k_1 \geq \dots. \quad (4.3)$$

This assumption is quite reasonable and is valid for phase functions in practical use. More assumptions on these numbers will be introduced wherever needed.

Let  $j = 1$ . We have

$$S_1 Y_n(\omega) = k_{1,n} Y_n(\omega), \quad k_{1,n} = \frac{\lambda_1}{1 + \alpha_1 n(n+1)}. \quad (4.4)$$

Equate the first two eigenvalues of  $S$  and  $S_1$ :

$$\lambda_1 = k_0, \quad \frac{\lambda_1}{1 + 2\alpha_1} = k_1.$$

Thus,

$$\lambda_1 = k_0, \quad \alpha_1 = \frac{1}{2} \left( \frac{k_0}{k_1} - 1 \right). \quad (4.5)$$

Observe that (4.1) and (4.2) are satisfied.

Then take  $j = 2$ ,

$$S_2 = \lambda_1 (I - \alpha_1 \Delta^*)^{-1} + \lambda_2 (I - \alpha_2 \Delta^*)^{-1} \quad (4.6)$$

with the parameters satisfying

$$\lambda_1 > 0, \quad \lambda_2 > 0, \quad \alpha_1 > 0, \quad \alpha_2 > 0, \quad \alpha_1 \neq \alpha_2. \quad (4.7)$$

We have

$$S_2 Y_n(\omega) = k_{2,n} Y_n(\omega), \quad k_{2,n} = \frac{\lambda_1}{1 + \alpha_1 n(n+1)} + \frac{\lambda_2}{1 + \alpha_2 n(n+1)}. \quad (4.8)$$



Require the parameters to match the first three eigenvalues

$$k_{2,0} = k_0, \quad k_{2,1} = k_1, \quad k_{2,2} = k_2,$$

i.e.,

$$\lambda_1 + \lambda_2 = k_0, \tag{4.9}$$

$$\frac{\lambda_1}{1 + 2\alpha_1} + \frac{\lambda_2}{1 + 2\alpha_2} = k_1, \tag{4.10}$$

$$\frac{\lambda_1}{1 + 6\alpha_1} + \frac{\lambda_2}{1 + 6\alpha_2} = k_2. \tag{4.11}$$

Consider the system (4.9)–(4.11) for a general form solution. Use  $\alpha_1$  as the parameter for the solution. Multiply (4.10) by  $(1 + 2\alpha_1)$  and subtract (4.9) from the resulting equation to obtain

$$\frac{2(\alpha_1 - \alpha_2)}{1 + 2\alpha_2} \lambda_2 = (k_1 - k_0) + 2k_1\alpha_1. \tag{4.12}$$

Multiply (4.11) by  $(1 + 6\alpha_1)$  and subtract (4.9) from the resulting equation to obtain

$$\frac{6(\alpha_1 - \alpha_2)}{1 + 6\alpha_2} \lambda_2 = (k_2 - k_0) + 6k_2\alpha_1. \tag{4.13}$$

Divide (4.12) from (4.13) to obtain

$$\frac{3(1 + 2\alpha_2)}{1 + 6\alpha_2} = \frac{(k_2 - k_0) + 6k_2\alpha_1}{(k_1 - k_0) + 2k_1\alpha_1}.$$

Noticing that the left side equals  $1 + 2/(1 + 6\alpha_2)$ , we find

$$\frac{1 + 6\alpha_2}{2} = \frac{(k_1 - k_0) + 2k_1\alpha_1}{(k_2 - k_1) + 2(3k_2 - k_1)\alpha_1}. \tag{4.14}$$

Thus,

$$\alpha_2 = \frac{1}{6} \cdot \frac{(3k_1 - 2k_0 - k_2) + 6(k_1 - k_2)\alpha_1}{(k_2 - k_1) + 2(3k_2 - k_1)\alpha_1}. \tag{4.15}$$

Then we use (4.13) to get

$$\lambda_2 = \frac{2[(k_1 - k_0) + 2k_1\alpha_1][(k_2 - k_0) + 6k_2\alpha_1]}{(2k_0 + k_2 - 3k_1) + 12(k_2 - k_1)\alpha_1 + 12(3k_2 - k_1)\alpha_1^2}. \tag{4.16}$$

Finally, from (4.12),

$$\lambda_1 = 1 - \lambda_2. \tag{4.17}$$

We now investigate the issue of positivity of the one parameter solution  $(\alpha_1, \alpha_2, \lambda_1, \lambda_2)$  given by the formulas (4.15)–(4.17), with  $\alpha_1 > 0$ . Since the differential approximation is intended mainly for applications with highly forward peaked scattering, here we assume

$$3k_2 > k_1, \quad 3k_0k_2 - k_0k_1 - 2k_1k_2 > 0. \quad (4.18)$$

The denominator of  $\lambda_2$  can be rewritten as

$$12(3k_2 - k_1) \left( \alpha_1 + \frac{k_2 - k_1}{2(3k_2 - k_1)} \right)^2 + \frac{2}{3k_2 - k_1} (3k_0k_2 - k_0k_1 - 2k_1k_2),$$

which is obviously positive under the assumption (4.18).

To have  $\lambda_2 > 0$ , we need the two factors in the numerator be of the same sign. This amounts to requiring

$$\alpha_1 > \max \left\{ \frac{k_0 - k_1}{2k_1}, \frac{k_0 - k_2}{6k_2} \right\} \quad \text{or} \quad \alpha_1 < \min \left\{ \frac{k_0 - k_1}{2k_1}, \frac{k_0 - k_2}{6k_2} \right\}.$$

Under the condition (4.18),

$$\frac{k_0 - k_1}{2k_1} > \frac{k_0 - k_2}{6k_2}.$$

So the condition for  $\lambda_2 > 0$  is

$$\alpha_1 > \frac{k_0 - k_1}{2k_1} \quad \text{or} \quad 0 < \alpha_1 < \frac{k_0 - k_2}{6k_2}. \quad (4.19)$$

To have  $\alpha_2 > 0$ , we need its numerator and denominator be of the same sign. This amounts to requiring

$$\alpha_1 > \max \left\{ \frac{2k_0 + k_2 - 3k_1}{6(k_1 - k_2)}, \frac{k_1 - k_2}{2(3k_2 - k_1)} \right\} \quad \text{or} \\ \alpha_1 < \min \left\{ \frac{2k_0 + k_2 - 3k_1}{6(k_1 - k_2)}, \frac{k_1 - k_2}{2(3k_2 - k_1)} \right\}.$$

Under the condition (4.18),

$$\frac{2k_0 + k_2 - 3k_1}{6(k_1 - k_2)} > \frac{k_1 - k_2}{2(3k_2 - k_1)}.$$

So the condition for  $\lambda_2 > 0$  is

$$\alpha_1 > \frac{2k_0 + k_2 - 3k_1}{6(k_1 - k_2)} \quad \text{or} \quad 0 < \alpha_1 < \frac{k_1 - k_2}{2(3k_2 - k_1)}. \quad (4.20)$$

Under the condition (4.18),

$$\frac{2k_0 + k_2 - 3k_1}{6(k_1 - k_2)} > \frac{k_0 - k_1}{2k_1}, \quad \frac{k_1 - k_2}{2(3k_2 - k_1)} < \frac{k_0 - k_2}{6k_2}.$$

Combining (4.19) and (4.20), we conclude that a condition for  $\alpha_1 > 0, \alpha_2 > 0$  and  $\lambda_2 > 0$  is

$$\alpha_1 > \frac{2k_0 + k_2 - 3k_1}{6(k_1 - k_2)} \quad \text{or} \quad 0 < \alpha_1 < \frac{k_1 - k_2}{2(3k_2 - k_1)}. \tag{4.21}$$

For  $\lambda_1 > 0$ , we need  $\lambda_2 < 1$ . This issue can usually be easily addressed for concrete phase functions, and we consider such an example in the next section.

### 5 An example

In the following demonstration, we take the Henyey–Greenstein phase function as an example; in this case,

$$k_n = g^n, \quad n = 0, 1, \dots$$

We use the operators  $\{S_j\}_{j \geq 1}$  of (3.9) to approximate  $S$ . To maintain ellipticity of the differential operator  $(I - \alpha_i \Delta^*)$ , we require

$$\alpha_i > 0, \quad 1 \leq i \leq j. \tag{5.1}$$

Moreover, for the benefit of stable numerical approximation for the operator  $S_j$ , it is desirable to have

$$\lambda_i > 0, \quad 1 \leq i \leq j, \tag{5.2}$$

wherever possible.

For the one term approximation  $S_1 = \lambda_1(I - \alpha_1 \Delta^*)^{-1}$ , from (4.5) we have

$$\lambda_1 = 1, \quad \alpha_1 = \frac{1 - g}{2g}. \tag{5.3}$$

For the two term approximation  $S_2 = \lambda_1(I - \alpha_1 \Delta^*)^{-1} + \lambda_2(I - \alpha_2 \Delta^*)^{-1}$ , by (4.15),

$$\alpha_2 = \frac{1 - g}{6g} \cdot \frac{g - 2 + 6g\alpha_1}{g - 1 + 2(3g - 1)\alpha_1}, \tag{5.4}$$

by (4.16),

$$\lambda_2 = \frac{2(g - 1 + 2g\alpha_1)(g^2 - 1 + 6g^2\alpha_1)}{(1 - g)(2 - g) + 12g(g - 1)\alpha_1 + 12g(3g - 1)\alpha_1^2}, \tag{5.5}$$

and from (4.17),

$$\lambda_1 = \frac{g(1-g)(2g-1)(1+8\alpha_1+12\alpha_1^2)}{(1-g)(2-g)+12g(g-1)\alpha_1+12g(3g-1)\alpha_1^2}. \quad (5.6)$$

We now investigate the issue of positivity of the one parameter solution  $(\alpha_1, \alpha_2, \lambda_1, \lambda_2)$  given by the formulas (5.4)–(5.6), with  $\alpha_1 > 0$ . The condition (4.18) is equivalent to

$$g > \frac{1}{2}, \quad (5.7)$$

a condition naturally valid in applications with highly forward peaked scattering. Henceforth, we always assume (5.7). Since (4.18) holds, the common denominator of  $\lambda_1$  and  $\lambda_2$  is positive and easily,  $\lambda_1 > 0$ . Thus, by (4.21), the condition for a positive solution  $(\alpha_1, \alpha_2, \lambda_1, \lambda_2)$  is

$$\alpha_1 > \frac{2-g}{6g} \quad \text{or} \quad 0 < \alpha_1 < \frac{1-g}{2(3g-1)}.$$

By the symmetry of  $S_2$  with respect to  $(\alpha_1, \lambda_1)$  and  $(\alpha_2, \lambda_2)$ , we have the equivalence

$$\begin{aligned} \alpha_1 > \frac{2-g}{6g} &\iff 0 < \alpha_2 < \frac{1-g}{2(3g-1)}, \\ 0 < \alpha_1 < \frac{1-g}{2(3g-1)} &\iff \alpha_2 > \frac{2-g}{6g}. \end{aligned}$$

In conclusion, in the case  $g > 1/2$ , the condition for a positive solution  $(\alpha_1, \alpha_2, \lambda_1, \lambda_2)$  is

$$\alpha_1 > \frac{2-g}{6g}. \quad (5.8)$$

Since  $\alpha_1 = 1/2$  satisfies (5.8), one solution is

$$\alpha_1 = \frac{1}{2}, \quad \alpha_2 = \frac{1-g}{6g}, \quad \lambda_1 = \frac{4g(1-g)}{4g-1}, \quad \lambda_2 = \frac{4g^2-1}{4g-1}. \quad (5.9)$$

## 6 Concluding remarks

In this paper, we study a family of differential approximation models for the radiative transfer equation (RTE), especially in the case of highly forward-peaked scattering. For the approximation models, the scattering operator in RTE operators is approximated by operators constructed from the Laplace–Beltrami operator such that the eigenvalues of the scattering operator and those of the approximation operators have the same limiting behavior and the first several eigenvalues are pairwise equal.

After introducing a general framework for the family of differential approximation models, we provide a detailed discussion on conditions for positivity of the parameters in the one-term and two-term approximation models. Closeness of the approximation models to the RTE can be illustrated through comparing eigenvalues of the integral operator in the RTE with those of the approximate operators in the approximation models.

Rigorous mathematical analysis and numerical analysis of the differential approximation models, numerical simulation results on closeness of solutions of the differential approximation models to the solution of the RTE, as well as related inverse problems and their applications, will be given in sequels. In particular, it will be shown that numerical simulation of the differential approximation models can be carried out substantially simpler than that for the RTE.

**Acknowledgements** This work was partially supported by NIH/NHLBI (R01 HL098912).

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